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The deformation of a drop in a general time-dependent fluid flow

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A theoretical method is given for the determination of the shape of a fluid drop in steady and unsteady flows by making an expansion in terms of the drop deformation. Effects of fluid viscosity and interfacial tension are taken into account. Examples given include the determination of the shape of a drop in shear and in hyperbolic flow when each is started impulsively from rest.

1. Introduction

Taylor (1934) discussed the behaviour of a fluid drop in shear flow, when the effects of fluid viscosity and interfacial tension are taken into account. In considering a drop of fluid of viscosity μ^* suspended in a fluid of viscosity μ_0 undergoing a shear flow of magnitude G, one may note that the behaviour of the drop must depend upon the two dimensionless parameters $\lambda = \mu^*/\mu_0$ and $k = \sigma/\mu_0 Ga$, where a is the radius of the drop and σ the interfacial tension. Taylor considered the case of a drop for which interfacial tension effects were dominant over viscous effects, i.e. the case for which $\lambda = O(1)$ and $k \gg 1$, and obtained the drop deformation to order k^{-1}. The case for which interfacial tension effects were negligible in comparison with viscous effects, i.e. $k = O(1)$ and $\lambda \gg 1$, was also considered; the drop deformation was obtained to order λ^{-1}. The results obtained for the former case were used by Chaffey, Brenner & Mason (1965*a*, *b*) to determine the migration of liquid drops in a shear flow near a plane wall.

It should be noted that for the interfacial tension dominated case, Taylor showed that the drop would deform into a spheroid with its major axis at an angle of 45° to the flow, whereas for the viscosity dominated case, it would deform into a spheroid with major axis in the direction of the flow. Thus it is seen that one cannot expect either of Taylor's results to apply to the case in which k and λ are both large and of the same order of magnitude. It is to be expected, however, that for such a case the drop deformation would be small; and this suggests that the case is perhaps amenable to theoretical investigation.

In the present paper drop deformation is assumed to be small and of order ϵ, where $\epsilon \ll 1$. Expansions of velocity fields are then made in terms of the parameter ϵ, no restrictions being placed upon λ and k other than those which may be implied by the assumption that deformation is small. Thus, rather than make expansions in either k^{-1} or λ^{-1}, one makes expansions in ϵ, so that all problems involving small drop deformation are solved simultaneously.

It is found that by this method the problem of the determination of the shape of a drop in a general time dependent flow is no more difficult than the determination of the shape of a drop placed in a steady shear. Hence, in §§ 2 to 5 of this paper, we shall be concerned with the determination of the deformation of a drop placed in a general time dependent flow. Then, in §§ 6 and 7, the general results obtained will be used to examine the behaviour of a drop in shear and hyperbolic flow when each of these is started impulsively from rest. The equilibrium shapes of a drop in steady shear flow are also obtained and are shown to agree with Taylor's (1934) results for the case in which $\lambda = O(1)$, $k \gg 1$, and the case in which $k = O(1)$, $\lambda \gg 1$. Agreement is also obtained with the experimental results of Rumscheidt & Mason (1961).

2. Method of expansion

A neutrally buoyant drop of a fluid of viscosity μ^* is suspended in a fluid of viscosity μ_0 which is undergoing a motion that deforms the drop from its equilibrium, spherical shape. We assume that the radius a of the undeformed drop is very much smaller than the length scale L of the basic fluid flow causing the drop deformation. If we define V as a characteristic fluid velocity in the neighbourhood of the drop (relative to its centre), then we assume that both the Reynolds numbers based upon a, V and the kinematic viscosities of the two fluids are so small that one may neglect inertia effects in the neighbourhood of the drop. We shall not assume, however, that the Reynolds number based upon the length scale L is small. Thus, on this large scale, the basic undisturbed fluid flow field will satisfy the full Navier–Stokes equations. Throughout the paper, we shall use quantities which have been made dimensionless by the viscosity μ_0, the length a and the velocity V, unless we state otherwise.

In the neighbourhood of the drop, where inertia effects are negligible, the undisturbed velocity of the suspending fluid $\mathbf{U}$ must satisfy the creeping motion equations

$$\left.\begin{aligned} \nabla^2\mathbf{U} - \nabla P &= \mathbf{0}, \\ \nabla . \mathbf{U} &= 0, \end{aligned}\right\} \tag{2.1}$$

where P is the corresponding pressure. Since $\mathbf{U}$ cannot possess any singularity at the origin, the solution of the equations (2.1) may be written in the form derived by Lamb (1932) as

$$\mathbf{U} = \sum_{n=0}^{\infty} \left[\nabla\chi_n \times \mathbf{r} + \nabla\phi_n + \frac{n+3}{2(n+1)(2n+3)} r^2 \nabla p_n - \frac{n}{(n+1)(2n+3)} \mathbf{r} p_n \right],$$

$$P = \sum_{n=0}^{\infty} p_n, \tag{2.2}$$

where χ_n, ϕ_n and p_n are solid spherical harmonics of order n. Since the radius a of the drop is very much smaller than the length scale L of this undisturbed velocity flow field, all terms in the above expression for $\mathbf{U}$, which behave like r^{+n} where $n > 1$, may be neglected. Also, since our axes move with the drop, and since the total hydrodynamic force on the drop resulting from the fluid flow must be zero, it follows that there cannot be any term like r^0 in the expression for **U**. Thus the velocity field **U** contains only terms which behave like r^{+1}, and so

$$\left.\begin{aligned} \chi_n &= 0, \quad \text{if} \quad n \neq 1, \\ \phi_n &= 0, \quad \text{if} \quad n \neq 2, \\ p_n &= 0, \quad \text{for all } n. \end{aligned}\right\} \tag{2.3}$$

Hence
$$\mathbf{U} = \nabla\chi_1 \times \mathbf{r} + \nabla\phi_2, \tag{2.4}$$
where
$$\chi_1 = rQ_1, \quad \phi_2 = r^2 S_2, \tag{2.5}$$

Q_1 and S_2 being time-dependent surface harmonics of orders one and two respectively. The vorticity $\boldsymbol{\omega}$ and rate-of-strain tensor e_{ij} of this flow field (2.4) are functions of time only and are given by

$$\omega_i = 2(rQ_1)_{,i}, \tag{2.6}$$

$$e_{ij} = (r^2 S_2)_{,ij}. \tag{2.7}$$

The undisturbed flow field **U** may then be written in the form

$$U_i = \tfrac{1}{2}\epsilon_{ijk}\omega_j r_k + e_{ij} r_j, \tag{2.8}$$

where $e_{kk} = 0$.

For a drop in the flow field (2.4), the deformation is determined by the two dimensionless parameters $\lambda = \mu^*/\mu_0$ and $k = \sigma/\mu_0 V$. The values of these parameters are assumed to be such that the drop deformation is small.

By taking the centre of the undeformed drop to be the origin of axes, the dimensionless position vector of a general point may be written as $\mathbf{r} = (r_1, r_2, r_3)$. Thus the equation of the surface of the drop in the deformed state may be represented by

$$r = 1 + \epsilon f(r_1/r, r_2/r, r_3/r), \tag{2.9}$$

where $r = (r_i r_i)^{\frac{1}{2}}$, ϵ being much smaller than unity. It is in terms of this parameter ϵ, that the expansions will be made, rather than in terms of the parameters λ^{-1} and k^{-1}. The only restrictions to be placed on λ and k are those which are implied by the condition that the drop deformation ϵ be small.

The flow fields inside and outside the *undeformed* drop $r = 1$ are defined to be $\mathbf{u}_0^*$ and $\bar{\mathbf{u}}_0$ respectively, and like **U** must also satisfy (2.1). Hence they are given by Lamb's (1932) solution as

$$\mathbf{u}_0^* = \sum_{n=0}^{\infty}\left[\nabla\chi_n^* \times \mathbf{r} + \nabla\phi_n^* + \frac{n+3}{2(n+1)(2n+3)} r^2 \nabla p_n^* - \frac{n}{(n+1)(2n+3)} \mathbf{r} p_n^*\right], \tag{2.10}$$

and

$$\bar{\mathbf{u}}_0 = \nabla\chi_1 \times \mathbf{r} + \nabla\phi_2 + \sum_{n=0}^{\infty}\left[\nabla\bar{\chi}_{-n-1} \times \mathbf{r} + \nabla\bar{\phi}_{-n-1} - \frac{n-2}{(2n)(2n-1)} r^2 \nabla \bar{p}_{-n-1} + \frac{n+1}{n(2n-1)} \mathbf{r}\bar{p}_{-n-1}\right], \tag{2.11}$$

where χ_n^*, ϕ_n^* and p_n^* are solid spherical harmonics of order n and $\bar{\chi}_{-n-1}$, $\bar{\phi}_{-n-1}$ and $\bar{p}_{-n-1}$ are solid spherical harmonics of order $-n-1$. The values of these harmonics are determined by the following boundary conditions on $r = 1$: the normal components of $\mathbf{u}_0^*$ and $\bar{\mathbf{u}}_0$ are zero, their tangential components are continuous, and the corresponding tangential stresses are also continuous, i.e.

$$\mathbf{u}_0^*.\mathbf{r} = \bar{\mathbf{u}}.\mathbf{r} = 0, \tag{2.12}$$

$$(\mathbf{u}_0^* - \mathbf{u}_0^*.\mathbf{rr}) = (\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_0.\mathbf{rr},) \tag{2.13}$$

$$\lambda(\mathbf{p}_0^*.\mathbf{r} - \mathbf{r}.\mathbf{p}_0^*.\mathbf{rr}) = (\bar{\mathbf{p}}_0.\mathbf{r} - \mathbf{r}.\bar{\mathbf{p}}_0.\mathbf{rr}), \tag{2.14}$$

where $\mathbf{p}_0^*$ and $\bar{\mathbf{p}}_0$ are the 'stress tensors' corresponding to the velocity fields $\mathbf{u}_0^*$ and $\bar{\mathbf{u}}_0$ respectively, i.e.

$$(p_0^*)_{ij} = -p_0^*\delta_{ij} + \{(u_0^*)_{i,j} + (u_0^*)_{j,i}\},$$

$$(\bar{p}_0)_{ij} = -\bar{p}_0\delta_{ij} + \{(\bar{u}_0)_{i,j} + (\bar{u}_0)_{j,i}\}.$$

In order to make $(u_0^*)_i$ and $(p_0^*)_{ij}$ of the same order of magnitude (even if λ is large), we have defined the dimensionless stress tensor $(p_0^*)_{ij}$ in terms of internal viscosity μ^*, so that relative to viscosity μ_0 the true stress tensor within the drop is $\lambda(p_0^*)_{ij}$.

The flow fields $\mathbf{u}^*$ and $\bar{\mathbf{u}}$ inside and outside the deformed drop are assumed to possess expansions in terms of the parameter ϵ, which are of the form

$$\left.\begin{aligned} \mathbf{u}^* &= \mathbf{u}_0^* + \epsilon\mathbf{u}_1^* + \ldots, \\ \bar{\mathbf{u}} &= \bar{\mathbf{u}}_0 + \epsilon\bar{\mathbf{u}}_1 + \ldots. \end{aligned}\right\} \tag{2.15}$$

Since $\mathbf{u}^*$, $\mathbf{u}_0^*$, $\bar{\mathbf{u}}$, $\bar{\mathbf{u}}_0$ all satisfy the creeping motion equation, so must the velocity fields $\mathbf{u}_1^*$ and $\bar{\mathbf{u}}_1$. The boundary conditions for $\mathbf{u}_1^*$ and $\bar{\mathbf{u}}_1$ are derived from the conditions of continuity of tangential and normal velocity and of tangential stress on the deformed surface; the difference in normal stress across the surface is balanced by interfacial tension forces. All terms of order ϵ^2 will be neglected.

3. Flow in undeformed drop

The fluid velocity inside and outside an undeformed drop $\mathbf{u}_0^*$ and $\bar{\mathbf{u}}_0$ is given by (2.10) and (2.11), in which the harmonic functions χ_n^*, ϕ_n^*, p_n^*, $\bar{\chi}_{-n-1}$, $\bar{\phi}_{-n-1}$ and $\bar{p}_{-n-1}$ are determined by the boundary conditions (2.12), (2.13) and (2.14). Corresponding to the velocity fields $\mathbf{u}_0^*$ and $\bar{\mathbf{u}}_0$, one has stress tensors $(p_0^*)_{ij}$ and $(\bar{p}_0)_{ij}$ respectively, given by

$$(p_0^*)_{ij} = \sum_{n=0}^{\infty}\left[\epsilon_{ikl}(\chi_n^*)_{,kj}r_l + \epsilon_{jkl}(\chi_n^*)_{,ki}r_l + 2(\phi_n^*)_{,ij} + \frac{n+3}{(n+1)(2n+3)}r^2(p_n^*)_{,ij} + \frac{3}{(n+1)(2n+3)}\{r_i(p_n^*)_{,j} + r_j(p_n^*)_{,i}\} - \frac{2n^2+7n+3}{(n+1)(2n+3)}(p_n^*)\delta_{ij}\right] \tag{3.1}$$

$$(\bar{p}_0)_{ij} = 2(\phi_2)_{,ij} + \sum_{n=0}^{\infty}\left[\epsilon_{ikl}(\bar{\chi}_{-n-1})_{,kj}r_l + \epsilon_{jkl}(\bar{\chi}_{-n-1})_{,ki}r_l + 2(\bar{\phi}_{-n-1})_{,ij} - \frac{n-2}{n(2n-1)}r^2(\bar{p}_{-n-1})_{,ij} + \frac{3}{n(2n-1)}\{r_i(\bar{p}_{-n-1})_{,j} + r_j(\bar{p}_{-n-1})_{,i}\} - \frac{2n^2-3n-2}{n(2n-1)}(\bar{p}_{-n-1})\delta_{ij}\right]. \tag{3.2}$$

Suppose one lets

$$\chi_n^* = r^n Q_n^*, \quad \phi_n^* = r^n S_n^*, \quad p_n^* = r^n T_n^*, \tag{3.3}$$

and

$$\bar{\chi}_{-n-1} = r^{-n-1}\bar{Q}_n, \quad \bar{\phi}_{-n-1} = r^{-n-1}\bar{S}_n, \quad \bar{p}_{-n-1} = r^{-n-1}\bar{T}_n, \tag{3.4}$$

where Q_n^*, S_n^*, T_n^*, $\bar{Q}_n$, $\bar{S}_n$ and $\bar{T}_n$ are spherical surface harmonics of order n. Then one may write expressions for the tangential and normal components of the velocity fields $\mathbf{u}_0^*$ and $\bar{\mathbf{u}}_0$, and of the tangential and normal components of the corresponding stresses on the surface $r = 1$, in the form

$$\left.\begin{aligned}
(u_0^*)_i r_i &= \sum_{n=0}^{\infty}\left[nS_n^* + \frac{n}{2(2n+3)}T_n^*\right],\\
\{(u_0^*)_i - (u_0^*)_j r_j r_i\} &= \sum_{n=0}^{\infty}\left[\epsilon_{ijk}(Q_n^*)_{,j}r_k + (S_n^*)_{,i} + \frac{n+3}{2(n+1)(2n+3)}(T_n^*)_{,i}\right],\\
\{r_i(p_0^*)_{ij}r_j\} &= \sum_{n=0}^{\infty}\left[2n(n-1)S_n^* + \frac{n^3-4n-3}{(n+1)(2n+3)}T_n^*\right],\\
\{(p_0^*)_{ij}r_j - r_j(p_0^*)_{jk}r_k r_i\} &= \sum_{n=0}^{\infty}\left[(n-1)\epsilon_{ijk}(Q_n^*)_{,j}r_k + 2(n-1)(S_n^*)_{,i}\right.\\
&\qquad\left. + \frac{n(n+2)}{(n+1)(2n+3)}(T_n^*)_{,i}\right],
\end{aligned}\right\} \tag{3.5}$$

$$\left.\begin{aligned}
(\bar{u}_0)_i r_i &= 2S_2 + \sum_{n=0}^{\infty}\left[-(n+1)\bar{S}_n + \frac{n+1}{2(2n+1)}\bar{T}_n\right],\\
\{(\bar{u}_0)_i - (\bar{u}_0)_j r_j r_i\} &= \epsilon_{ijk}(Q_1)_{,j}r_k + (S_2)_{,i} + \sum_{n=0}^{\infty}\left[\epsilon_{ijk}(\bar{Q}_n)_{,j}r_k + (\bar{S}_n)_{,i}\right.\\
&\qquad\left. - \frac{n-2}{2n(2n-1)}(\bar{T}_n)_{,i}\right],\\
\{r_i(\bar{p}_0)_{ij}r_j\} &= 4S_2 + \sum_{n=0}^{\infty}\left[2(n+1)(n+2)\bar{S}_n - \frac{n^3+3n^2-n}{n(2n-1)}\bar{T}_n\right],\\
\{(\bar{p}_0)_{ij}r_j - r_j(\bar{p}_0)_{jk}r_k r_i\} &= 2(S_2)_{,i} + \sum_{n=0}^{\infty}\left[-(n+2)\epsilon_{ijk}(\bar{Q}_n)_{,j}r_k\right.\\
&\qquad\left. - 2(n+2)(\bar{S}_n)_{,i} + \frac{(n-1)(n+1)}{n(2n-1)}(\bar{T}_n)_{,i}\right].
\end{aligned}\right\} \tag{3.6}$$

Substituting these expressions into (2.12), (2.13) and (2.14) ,one obtains

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left[nS_n^* + \frac{n}{2(2n+3)}T_n^*\right] = 0,\\
&2S_2 + \sum_{n=0}^{\infty}\left[-(n+1)\bar{S}_n + \frac{n+1}{2(2n-1)}\bar{T}_n\right] = 0,\\
&\sum_{n=0}^{\infty}\left[\epsilon_{ijk}(Q_n^*)_{,j}r_k + (S_n^*)_{,i} + \frac{n+3}{2(n+1)(2n+3)}(T_n^*)_{,i}\right] = \epsilon_{ijk}(Q_1)_{,j}r_k + (S_2)_{,i}\\
&\qquad + \sum_{n=0}^{\infty}\left[\epsilon_{ijk}(\bar{Q}_n)_{,j}r_k + (\bar{S}_n)_{,i} - \frac{n-2}{2n(2n-1)}(\bar{T}_n)_{,i}\right],\\
&\sum_{n=0}^{\infty}\lambda\left[(n-1)\epsilon_{ijk}(Q_n^*)_{,j}r_k + 2(n-1)(S_n^*)_{,i} + \frac{n(n+2)}{(n+1)(2n+3)}(T_n^*)_{,i}\right] = 2(S_2)_{,i}\\
&\qquad + \sum_{n=0}^{\infty}\left[-(n+2)\epsilon_{ijk}(\bar{Q}_n)_{,j}r_k - 2(n+2)(\bar{S}_n)_{,i} + \frac{(n-1)(n+1)}{n(2n-1)}(\bar{T}_n)_{,i}\right].
\end{aligned} \tag{3.7}$$

The solution of this set of equations is

$$\left.\begin{aligned} &Q_n^* = \bar{Q}_n = 0, \quad \text{if} \quad n \neq 1, \\ &S_n^* = \bar{S}_n = \bar{T}_n = 0, \quad \text{if} \quad n \neq 2, \\ &T_n^* = 0, \quad \text{if} \quad n \neq 0, 2. \end{aligned}\right\} \tag{3.8}$$

The remaining surface harmonics, Q_1^*, $\bar{Q}_1$, S_2^*, $\bar{S}_2$, T_2^*, $\bar{T}_2$ and T_0^*, satisfy the following linear equations:

$$\left.\begin{aligned} 2S_2^* + \tfrac{1}{7}T_2^* &= 0, \\ 2S_2 - 3\bar{S}_2 + \tfrac{1}{2}\bar{T}_2 &= 0, \\ Q_1^* &= \bar{Q}_1 + Q_1, \\ S_2^* + \tfrac{5}{42}T_2^* &= \bar{S}_2 + S_2, \\ \bar{Q}_1 &= 0, \\ 2\lambda S_2^* + \tfrac{8}{21}\lambda T_2^* &= -8\bar{S}_2 + \tfrac{1}{2}\bar{T}_2 + 2S_2, \\ T_0^* \text{ arbitrary.}& \end{aligned}\right\} \tag{3.9}$$

This set of equations has the solution

$$\left.\begin{aligned} &Q_1^* = Q_1, \quad \bar{Q}_1 = 0, \\ &S_2^* = -\frac{3}{2(\lambda+1)}S_2, \quad \bar{S}_2 = -\frac{\lambda}{\lambda+1}S_2, \\ &T_2^* = +\frac{21}{\lambda+1}S_2, \quad \bar{T}_2 = -\frac{2(5\lambda+2)}{\lambda+1}S_2, \\ &T_0^* \text{ arbitrary.} \end{aligned}\right\} \tag{3.10}$$

Hence, by substituting these values into (3.3) and (3.4), one obtains the following velocity fields for $\mathbf{u}_0^*$ and $\bar{\mathbf{u}}_0$ from (2.10) and (2.11):

$$\left.\begin{aligned} \mathbf{u}_0^* &= \left[\nabla(rQ_1) \times \mathbf{r} - \frac{3}{2(\lambda+1)}\nabla(r^2S_2) + \frac{5}{2(\lambda+1)}r^2\nabla(r^2S_2) - \frac{2}{\lambda+1}\mathbf{r}(r^2S_2)\right], \\ \bar{\mathbf{u}}_0 &= [\nabla(rQ_1) \times \mathbf{r} + \nabla(r^2S_2)] + \left[-\frac{\lambda}{\lambda+1}\nabla(r^{-3}S_2) - \frac{5\lambda+2}{\lambda+1}\mathbf{r}(r^{-3}S_2)\right]. \end{aligned}\right\} \tag{3.11}$$

The arbitrary value of T_0^* does not contribute to the value of $\mathbf{u}_0^*$ and represents merely an arbitrary constant pressure within the drop.

It should be noticed that, for the special case of a drop in a plane shear flow given by

$$\mathbf{U} = (0, 0, \gamma r_2), \tag{3.12}$$

the values of Q_1 and S_2 are

$$Q_1 = \tfrac{1}{2}\gamma r_1 r^{-1}, \quad S_2 = \tfrac{1}{2}\gamma r_2 r_3 r^{-2}. \tag{3.13}$$

Thus the velocity fields inside and outside the drop are given by

$$\left.\begin{aligned} (u_0^*)_i &= \frac{\gamma}{4(\lambda+1)}\{(5r^2 + 2\lambda - 1)\, r_2\delta_{3i} + (5r^2 - 2\lambda - 5)\, r_3\delta_{2i} - 4r_2r_3r_i\}, \\ (\bar{u}_0)_i &= \frac{\gamma}{2(\lambda+1)}\{-\lambda r^{-5}(r_2\delta_{3i} + r_3\delta_{2i}) + r^{-5}r_2r_3r_i(5\lambda r^{-2} - 5\lambda - 2)\} + \gamma r_2\delta_{3i}. \end{aligned}\right\} \tag{3.14}$$

This agrees with the solution given by Bartok & Mason (1958), which was derived from Taylor's (1932) solution for a drop suspended in a fluid undergoing a hyperbolic flow.

4. Boundary conditions for $\mathbf{u}_1^*$ and $\bar{\mathbf{u}}_1$

Having obtained the velocity fields $\mathbf{u}_0^*$ and $\bar{\mathbf{u}}_0$ in § 3, we now derive the boundary conditions which the first-order velocity fields $\mathbf{u}_1^*$ and $\bar{\mathbf{u}}_1$ must satisfy. As stated in § 2, one requires to order ϵ (i) that the normal components of $\mathbf{u}^*$ and $\bar{\mathbf{u}}$ are continuous, (ii) that the tangential components of $\mathbf{u}^*$ and $\bar{\mathbf{u}}$ are continuous, (iii) that the tangential components of the corresponding stresses are continuous, and (iv) that the difference in the normal components of the stresses are balanced by interfacial tension forces, on the *deformed* surface

$$r = 1+\epsilon f(r_i/r). \tag{4.1}$$

The value of $\mathbf{u}_0^*$ on the surface is given in terms of its value on $r = 1$ by the relation

$$[(u_0^*)_i]_{r=1+\epsilon f} = [(u_0^*)_i+\epsilon f r_j (u_0^*)_{i,j}]_{r=1}+O(\epsilon^2). \tag{4.2}$$

Hence, since $\mathbf{u}^*$ is given by (2.15), it follows that

$$[(u^*)_i]_{r=1+\epsilon f} = [(u_0^*)_i+\epsilon f r_j (u_0^*)_{i,j}+\epsilon (u_1^*)_i]_{r=1}+O(\epsilon^2). \tag{4.3}$$

In a similar manner, it may be shown that the value of the stress tensor $(p^*)_{ij}$ corresponding to the velocity field $\mathbf{u}^*$, has on the *deformed* surface a value given by

$$[(p^*)_{ij}]_{r=1+\epsilon f} = [(p_0^*)_{ij}+\epsilon f r_k (p_0^*)_{ij,k}+\epsilon (p_1^*)_{ij}]_{r=1}+O(\epsilon^2). \tag{4.4}$$

The unit normal $\mathbf{n}$ to the surface (4.1) is in the direction of the gradient of $(r-1-\epsilon f)$. Thus

$$n_i = K(r_1 r^{-1}-\epsilon f_{,i}),$$

where K is a constant. Since $n_i n_i = 1$, it follows that

$$K^2(r_i r^{-1}-\epsilon f_{,i})(r_i r^{-1}-\epsilon f_{,i}) = 1,$$

$$K^2(1+\epsilon^2 f_{,i} f_{,i}) = 1,$$

or

$$K = 1+O(\epsilon^2).$$

Therefore the unit normal $\mathbf{n}$ is given by

$$n_i = r_i r^{-1}-\epsilon f_{,i}+O(\epsilon^2). \tag{4.5}$$

Consider a point P on the deformed surface (4.1) and take (ξ, η, ζ) Cartesian axes with P as origin and with the ζ axis in the direction of the normal $\mathbf{n}$ to the surface. The ξ and η axes are chosen in such a manner that the $(\xi\zeta)$ and $(\eta\zeta)$ planes are the principal planes of curvature of the surface at the point P. Let R_1 and R_2 be the corresponding principal radii of curvature. Then it may be shown that

$$\left.\begin{aligned} R_1 &= 1+\epsilon\left(f+\frac{\partial^2 f}{\partial \xi^2}\right)+O(\epsilon^2), \\ R_2 &= 1+\epsilon\left(f+\frac{\partial^2 f}{\partial \eta^2}\right)+O(\epsilon^2). \end{aligned}\right\} \tag{4.6}$$

Hence the sum of the principal curvatures is

$$\frac{1}{R_1}+\frac{1}{R_2} = 2-\epsilon\left(2f+\frac{\partial^2 f}{\partial \xi^2}+\frac{\partial^2 f}{\partial \eta^2}\right)+O(\epsilon^2). \tag{4.7}$$

However, since f is a function only of r_i/r, it follows that derivatives of f in the radial direction are zero. Hence it is seen that

$$\partial^2 f/\partial \zeta^2 = O(\epsilon),$$

so that
$$\frac{1}{R_1}+\frac{1}{R_2} = 2-\epsilon\left(2f+\frac{\partial^2 f}{\partial \xi^2}+\frac{\partial^2 f}{\partial \eta^2}+\frac{\partial^2 f}{\partial \zeta^2}\right);$$

but since the quantity
$$\left(\frac{\partial^2 f}{\partial \xi^2}+\frac{\partial^2 f}{\partial \eta^2}+\frac{\partial^2 f}{\partial \zeta^2}\right)$$

is invariant under an orthogonal transformation we may write this as $f_{,kk}$. Thus

$$\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = 2-\epsilon(2f+f_{,kk}). \tag{4.8}$$

By making use of (4.3), (4.4) and (4.5), one may obtain values for the normal and tangential components of velocity $\mathbf{u}^*$ and stress $(p^*)_{ij}$ in the form

$$\left.\begin{aligned}
&[(u^*)_i n_i]_{r=1+\epsilon f} = [(u_0^*)_i r_i + \epsilon f r_i r_j (u_0^*)_{i,j} + \epsilon (u_1^*)_i r_i - \epsilon f_{,i} (u_0^*)_i]_{r=1},\\
&[(u^*)_i - (u^*)_j n_j n_i]_{r=1+\epsilon f} = [(u_0^*)_i - (u_0^*)_j r_j r_i + \epsilon f r_j (u_0^*)_{i,j} + \epsilon (u_1^*)_i\\
&\qquad - \epsilon f r_i r_j r_k (u_0^*)_{j,k} - \epsilon (u_1^*)_j r_j r_i + \epsilon f_{,i} r_j (u_0^*)_j + \epsilon f_{,j} r_i (u_0^*)_j]_{r=1},\\
&[n_i (p^*)_{ij} n_j]_{r=1+\epsilon f} = [r_i r_j (p_0^*)_{ij} + \epsilon f r_i r_j r_k (p_0^*)_{ij,k} + \epsilon r_i r_j (p_1^*)_{ij}\\
&\qquad - \epsilon f_{,i} r_j (p_0^*)_{ij} - \epsilon f_{,j} r_i (p_0^*)_{ij}]_{r=1},\\
&[(p^*)_{ij} n_j - n_i n_j n_k (p^*)_{jk}]_{r=1+\epsilon f} = [(p_0^*)_{ij} r_j - r_i r_j r_k (p_0^*)_{jk}\\
&\qquad + \epsilon f r_j r_k (p_0^*)_{ij,k} + \epsilon r_j (p_1^*)_{ij} - \epsilon f_{,j} (p_0^*)_{ij}\\
&\qquad - \epsilon f r_i r_j r_k r_l (p_0^*)_{jk,l} - \epsilon r_i r_j r_k (p_1^*)_{jk} + \epsilon f_{,i} r_j r_k (p_0^*)_{jk}\\
&\qquad + \epsilon f_{,j} r_i r_k (p_0^*)_{jk} + \epsilon f_{,k} r_i r_j (p_0^*)_{jk}]_{r=1},
\end{aligned}\right\} \tag{4.9}$$

where terms of order ϵ^2 have been neglected. The expressions for the tangential and normal components of the velocity $\bar{\mathbf{u}}$ and stress $(\bar{p})_{ij}$ are exactly the expressions given in (4.9) with the quantities $(u_0^*)_i$, $(u_1^*)_i$, $(p_0^*)_{ij}$ and $(p_1^*)_{ij}$ replaced by $(\bar{u}_0)_i$, $(\bar{u}_1)_i$, $(\bar{p}_0)_{ij}$ and $(\bar{p}_1)_{ij}$ respectively. Thus one may now use these expressions to write down the boundary conditions (i) to (iv) given at the beginning of this section. Upon making use of the relations (2.12), (2.13), (2.14) and (4.8), one then obtains on $r = 1$,

$$\left.\begin{aligned}
\frac{\partial f}{\partial t} &= r_i (u_1^*)_i + f r_i r_j (u_0^*)_{i,j} - f_{,i} (u_0^*)_i,\\
&= r_i (\bar{u}_1)_i + f r_i r_j (\bar{u}_0)_{i,j} - f_{,i} (\bar{u}_0)_i,
\end{aligned}\right\} \tag{4.10}$$

$$\mathscr{T}[(u_1^*)_i + f r_j (u_0^*)_{i,j}] = \mathscr{T}[(\bar{u}_1)_i + f r_j (\bar{u}_0)_{i,j}], \tag{4.11}$$

$$\begin{aligned}
&-\lambda\{r_i r_j (p_0^*)_{ij}\} - \lambda\epsilon\{f r_i r_j r_k (p_0^*)_{ij,k} - f_{,j} r_i (p_0^*)_{ij} - f_{,i} r_j (p_0^*)_{ij}\\
&+ r_i r_j (p_1^*)_{ij}\} + \{r_i r_j (\bar{p}_0)_{ij}\} + \epsilon\{f r_i r_j r_k (\bar{p}_0)_{ij,k} - f_{,j} r_k (\bar{p}_0)_{ij}\\
&- f_{,i} r_j (\bar{p}_0)_{ij} + r_i r_j (\bar{p}_1)_{ij}\} = 2k - k\epsilon(2f + f_{,kk}),
\end{aligned} \tag{4.12}$$

$$\mathscr{T}[\lambda\{(p_1^*)_{ij}r_j+fr_jr_k(p_0^*)_{ij,k}-f_{,j}(p_0^*)_{ij}+f_{,i}r_jr_k(p_0^*)_{jk}\}]$$
$$=\mathscr{T}[(\overline{p}_1)_{ij}r_j+fr_jr_k(\overline{p}_0)_{ij,k}-f_{,j}(\overline{p}_0)_{ij}+f_{,i}r_jr_k(\overline{p}_0)_{jk}], \quad (4.13)$$

where $\mathscr{T}$ represents the operation of taking the tangential part, i.e.

$$\mathscr{T}[a_i] = a_i - a_j r_j r_i. \quad (4.14)$$

From (3.3) and (3.10) it is seen that in the expression (3.1) for $(p_0^*)_{ij}$, the only non-zero spherical harmonics χ_n^*, ϕ_n^* and p_n^* which appear are

$$\chi_1^* = rQ_1,$$
$$\phi_2^* = -\frac{3}{2(\lambda+1)}r^2S_2,$$
$$p_2^* = +\frac{21}{(\lambda+1)}r^2S_2.$$

Hence, since $(\chi_1^*)_{,kj} = 0$, it is seen that

$$(p_0^*)_{ij} = O(1/\lambda) \quad \text{as} \quad \lambda \to \infty. \quad (4.15)$$

In the normal stress boundary condition (4.12), terms of order unity will be retained, whilst terms of order ϵ are neglected. On the other hand, since no restriction has been placed on the quantities λ and k, terms in $(\lambda\epsilon)$ and $(k\epsilon)$ should be retained, because the parameters λ and k could be of order ϵ^{-1}. However terms in (4.12) like $\{-\lambda\epsilon f r_i r_j r_k (p_0^*)_{ij,k}\}$, which involve $(p_0^*)_{ij}$, are really of order ϵ^{+1}, since by (4.15) $(p_0^*)_{ij}$ is of order λ^{-1} if λ is large. Hence such terms may also be neglected. Thus, to order ϵ^0, (4.12) reduces to

$$-\lambda\{r_ir_j(p_0^*)_{ij}\}-\lambda\epsilon\{r_ir_j(p_1^*)_{ij}\}+\{r_ir_j(\overline{p}_0)_{ij}\} = 2k-k\epsilon(2f+f_{,kk}). \quad (4.16)$$

The values of $\{r_ir_j(p_0^*)_{ij}\}$ and $\{r_ir_j(\overline{p}_0)_{ij}\}$ are given by (3.5) and (3.6) with values of S_n^*, T_n^*, $\overline{S}_n$ and $\overline{T}_n$, as given in (3.10). Hence one may obtain

$$r_ir_j(p_0^*)_{ij} = 4S_2^* - \tfrac{1}{7}T_2^* - T_0^* = -\frac{9}{\lambda+1}S_2 - T_0^*, \quad (4.17)$$

$$r_ir_j(\overline{p}_0)_{ij} = 4S_2 + 24\overline{S}_2 - 3\overline{T}_2 = \frac{2(5\lambda+8)}{\lambda+1}S_2. \quad (4.18)$$

The term $\{-\lambda\epsilon r_i r_j (p_1^*)_{ij}\}$ appearing in (4.16) is of order unity only if λ is of order ϵ^{-1}. Hence, for the calculation of $(p_1^*)_{ij}$ in this expression, one may consider λ very large. Thus one may calculate $(p_1^*)_{ij}$ from the flow field $(u_1^*)_i$ derived from (4.10), (4.11) and (4.13), with $\lambda \to \infty$. In particular, the values of $(u_0^*)_i$ and $(\overline{u}_0)_i$ used in these equations may be taken to be given by (2.10) and (2.11) with the spherical harmonics, χ_n^*, ϕ_n^*, p_n^*, etc., given by (3.3) and (3.4), where

$$\left.\begin{aligned} Q_1^* = Q_1, \quad S_2^* = T_2^* = 0,\\ \overline{Q}_1 = 0, \quad \overline{S}_2 = -S_2, \quad \overline{T}_2 = -10S_2.\end{aligned}\right\} \quad (4.19)$$

Hence, evaluating $(u_0^*)_i$ and $(\overline{u}_0)_i$ in the limit $\lambda\to\infty$, and obtaining the corresponding stress tensors $(p_0^*)_{ij}$ and $(\overline{p}_0)_{ij}$, one may find the boundary conditions (4.10), (4.11) and (4.13) in the limit $\lambda\to\infty$, in the form

$$\partial f/\partial t = r_i(u_1^*)_i - f_{,i}\epsilon_{ijk}(Q_1)_{,j}r_k,$$
$$= r_i(\overline{u}_1)_i - f_{,i}\epsilon_{ijk}(Q_1)_{,j}r_k, \quad (4.20)$$

$$\mathscr{T}[(u_1^*)_i-(\bar{u}_1)_i] = \mathscr{T}[5f(S_2)_{,i}], \tag{4.21}$$

$$\mathscr{T}[(p_1^*)_{ij} r_j] = 0. \tag{4.22}$$

The boundary condition (4.16) on the normal stress reduces to

$$\lambda\epsilon\{r_i r_j (p_1^*)_{ij}\} = \frac{19\lambda+16}{\lambda+1} S_2 + \lambda T_0^* - 2k + k\epsilon(2f+f_{,kk}). \tag{4.23}$$

If the drop is in a fluid which is not moving, then

$$f = 0, \quad S_2 = Q_1 = 0, \quad (p_1^*)_{ij} = 0.$$

Hence
$$\lambda T_0^* - 2k = 0. \tag{4.24}$$

Substituting the value of T_0^* back into (4.23) yields

$$\lambda\epsilon\{r_i r_j (p_1^*)_{ij}\} = \frac{19\lambda+16}{\lambda+1} S_2 + k\epsilon(2f+f_{,kk}). \tag{4.25}$$

5. Drop deformation

The function $f(r_i/r)$ determining drop deformation may be expanded in the form

$$f = \sum_{n=2}^{\infty} F_n, \tag{5.1}$$

where F_n is a spherical surface harmonic of order n. Harmonics of order one and zero have been omitted, because they represent a translation and a dilatation of the drop. Using spherical polar axes (r, θ, ϕ) with origin at the centre of the drop, one may write

$$F_n = \sum_m A_{mn} P_n^m(\cos\theta)\, e^{im\phi}, \tag{5.2}$$

Now, since on $r = 1$

$$(F_n)_{,kk} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 F_n}{\partial\phi^2},$$

it may be shown by direct substitution that

$$(F_n)_{,kk} = -n(n+1)F_n.$$

Hence
$$f_{,kk} = -\sum_{n=2}^{\infty} n(n+1)F_n. \tag{5.3}$$

Since $\mathbf{u}_1^*$ and $\bar{\mathbf{u}}_1$ each satisfy the creeping motion equations they may be expanded in the forms

$$\left.\begin{aligned}
\mathbf{u}_1^* &= \sum_{n=0}^{\infty}\left[\nabla\tilde{\chi}_n^* \times \mathbf{r} + \nabla\tilde{\phi}_n^* + \frac{n+3}{2(n+1)(2n+3)} r^2 \nabla\tilde{p}_n^* - \frac{n}{(n+1)(2n+3)}\mathbf{r}\tilde{p}_n^*\right],\\
\bar{\mathbf{u}}_1 &= \sum_{n=0}^{\infty}\left[\nabla\tilde{\bar{\chi}}_{-n-1} \times \mathbf{r} + \nabla\tilde{\bar{\phi}}_{-n-1} - \frac{n-2}{(2n)(2n-1)} r^2 \nabla\tilde{\bar{p}}_{-n-1} - \frac{n+1}{n(2n-1)}\mathbf{r}\tilde{\bar{p}}_{-n-1}\right],
\end{aligned}\right\} \tag{5.4}$$

where $\tilde{\chi}_n^*$, $\tilde{\phi}_n^*$, $\tilde{p}_n^*$, $\tilde{\bar{\chi}}_n$, $\tilde{\bar{\phi}}_n$ and $\tilde{\bar{p}}_n$ are given by

$$\left.\begin{aligned}
&\tilde{\chi}_n^* = r^n \tilde{Q}_n^*, \quad \tilde{\phi}_n^* = r^n \tilde{S}_n^*, \quad \tilde{p}_n^* = r^n \tilde{T}_n^*,\\
&\tilde{\bar{\chi}}_{-n-1} = r^{-n-1}\tilde{\bar{Q}}_n, \quad \tilde{\bar{\phi}}_{-n-1} = r^{-n-1}\tilde{\bar{S}}_n, \quad \tilde{\bar{p}}_{-n-1} = r^{-n-1}\tilde{\bar{T}}_n,
\end{aligned}\right\} \tag{5.5}$$

$\tilde{Q}_n^*$, $\tilde{S}_n^*$, $\tilde{T}_n^*$, $\tilde{\tilde{Q}}_n$, $\tilde{\tilde{S}}_n$ and $\tilde{\tilde{T}}_n$ being spherical surface harmonics of order n. From the above values of $\mathbf{u}_1^*$ and $\bar{\mathbf{u}}_1$, one may obtain expressions for the corresponding stress tensors $(p_1^*)_{ij}$ and $(\bar{p}_1)_{ij}$, which can be substituted into the boundary conditions (4.20), (4.21), (4.22) and (4.25), to obtain

$$\sum_{n=2}^{\infty}\frac{\partial F_n}{\partial t} = \sum_{n=0}^{\infty}\left[n\tilde{S}_n^* + \frac{n}{2(2n+3)}\tilde{T}_n^*\right] - \left\{\sum_{n=2}^{\infty}(F_n)_{,i}\right\}\epsilon_{ijk}(Q_1)_{,j}r_k,$$
$$= \sum_{n=0}^{\infty}\left[-(n+1)\tilde{\tilde{S}}_n + \frac{n+1}{2(2n-1)}\tilde{\tilde{T}}_n\right] - \left\{\sum_{n=2}^{\infty}(F_n)_{,i}\right\}\epsilon_{ijk}(Q_1)_{,j}r_k, \quad (5.6)$$

$$\sum_{n=0}^{\infty}\left[\epsilon_{ijk}(\tilde{Q}_n)_{,j}r_k + (\tilde{S}_n^*)_{,i} + \frac{n+3}{2(n+1)(2n+3)}(\tilde{T}_n^*)_{,i} - \epsilon_{ijk}(\tilde{\tilde{Q}}_n)_{,j}r_k - (\tilde{\tilde{S}}_n)_{,i}\right.$$
$$\left. + \frac{n-2}{2n(2n-1)}(\tilde{\tilde{T}}_n)_{,i}\right] = 5(S_2)_{,i}\left\{\sum_{n=2}^{\infty}F_n\right\}, \quad (5.7)$$

$$\sum_{n=0}^{\infty}\left[(n-1)\,\epsilon_{ijk}(\tilde{Q}_n^*)_{,j}r_k + 2(n-1)\,(\tilde{S}_n^*)_{,i} + \frac{n(2n+2)}{(n+1)(2n+3)}(\tilde{T}_n^*)_{,i}\right] = 0, \quad (5.8)$$

$$\lambda\epsilon\sum_{n=0}^{\infty}\left[2n(n-1)\,(\tilde{S}_n^*) + \frac{n^3-4n-3}{(n+1)(2n+3)}(\tilde{T}_n^*)\right]$$
$$= \left(\frac{19\lambda+16}{\lambda+1}\right)S_2 + k\epsilon\left\{\sum_{n=2}^{\infty}(2-n^2-n)\,F_n\right\}, \quad (5.9)$$

where values of f and $f_{,kk}$ have been substituted using the relations (5.1) and (5.3).

From (5.8) and (5.9), it is seen that

$$2(n-1)\,\tilde{S}_n^* + \frac{n(n+2)}{(n+1)(2n+3)}\tilde{T}_n^* = 0, \quad \tilde{Q}_n^* = 0, \quad (5.10)$$

$$\lambda\epsilon\left\{2n(n-1)\,\tilde{S}_n^* + \frac{n^3-4n-3}{(n+1)(2n+3)}\tilde{T}_n^*\right\} = \left(\frac{19\lambda+16}{\lambda+1}\right)\delta_{2n}S_2 + k\epsilon(2-n-n^2)\,F_n$$

for all $n \geqslant 2$,

and that
$$\tilde{Q}_0^* = \tilde{S}_0^* = \tilde{T}_0^* = \tilde{T}_1^* = 0. \quad (5.11)$$

It may be shown that $\quad [r^n(F_n)_{,i}\epsilon_{ijk}(Q_1)_{,j}r_k]_{,ll} = 0,$

so the terms $\{(F_n)_{,i}\epsilon_{ijk}(Q_1)_{,j}r_k\}$, which appear in (5.6), are spherical surface harmonics of order n. Thus, from the first part of (5.6), it is observed that

$$\frac{\partial F_n}{\partial t} = n\tilde{S}_n^* + \frac{n}{2(2n+3)}\tilde{T}_n^* - (F_n)_{,i}\epsilon_{ijk}(Q_1)_{,j}r_k \quad \text{for all n} \geqslant 2, \quad (5.12)$$

and that
$$\tilde{S}_1^* + \tfrac{1}{10}\tilde{T}_1^* = 0. \quad (5.13)$$

Solving (5.10) for $\tilde{S}_n^*$ and $\tilde{T}_n^*$ ($n \geqslant 2$), one obtains

$$\left.\begin{aligned}\tilde{S}_n^* &= \frac{n(n+2)}{2(n-1)(2n^2+4n+3)}(\lambda\epsilon)^{-1}\left\{\left(\frac{19\lambda+16}{\lambda+1}\right)\delta_{2n}S_2 + k\epsilon(2-n-n^2)\,F_n\right\},\\ \tilde{T}_n^* &= -\frac{(n+1)(2n+3)}{2n^2+4n+3}(\lambda\epsilon)^{-1}\left\{\left(\frac{19\lambda+16}{\lambda+1}\right)\delta_{2n}S_2 + k\epsilon(2-n-n^2)\,F_n\right\}.\end{aligned}\right\} \quad (5.14)$$

Substitution of these values into (5.12) yields for all $n \geqslant 2$,

$$\frac{\partial F_n}{\partial t} = \frac{n(2n+1)}{2(2n^2+4n+3)(n-1)}(\lambda\epsilon)^{-1}\left\{\left(\frac{19\lambda+16}{\lambda+1}\right)\delta_{2n}S_2 + k\epsilon(2-n-n^2)F_n\right\} - (F_n)_{,i}\epsilon_{ijk}(Q_1)_{,j}r_k. \quad (5.15)$$

This gives the deformation of the drop as a function of time; the quantities S_2 and Q_1 are determined by the undisturbed fluid flow $\mathbf{U}$ through the relation

$$U_i = \epsilon_{ijk}(rQ_1)_{,j}r_k + (r^2S_2)_{,i}. \quad (5.16)$$

The vorticity $\boldsymbol{\omega}$ and rate-of-strain tensor e_{ij} corresponding to this flow field $\mathbf{U}$ is therefore given by

$$\omega_i = 2(rQ_1)_{,i}, \quad (5.17)$$

$$e_{ij} = (r^2S_2)_{,ij}. \quad (5.18)$$

The quantities F_n and S_2, being spherical surface harmonics, may be written in the form

$$F_n = F_{p_1p_2\ldots p_n}\left[\frac{\partial^n}{\partial r_{p_1}\partial r_{p_2}\ldots\partial r_{p_n}}(r^{-1})\right]r^{n+1}, \quad (5.19)$$

$$S_2 = S_{pq}\left[\frac{\partial^2}{\partial r_p\partial r_q}(r^{-1})\right]r^3, \quad (5.20)$$

where $F_{p_1p_2\ldots p_n}$ is an nth-order tensor and S_{pq} a second-order tensor. $F_{p_1p_2\ldots p_n}$ and S_{pq} may be taken to be symmetric with regard to interchange of indices. Also, since $(r^{-1})_{,pp} = 0$, one may take $S_{pp} = 0$. Thus the substitution of (5.20) into the expression (5.18) for e_{ij} gives

$$e_{ij} = 6S_{ij}. \quad (5.21)$$

By the use of (5.17) ,(5.15) may be reduced to

$$\frac{\partial F_n}{\partial t} + \tfrac{1}{2}\epsilon_{ijk}(F_n)_{,i}\omega_j r_k = \frac{n(2n+1)}{2(n-1)(2n^2+4n+3)}(\lambda\epsilon)^{-1}\left\{\left(\frac{19\lambda+16}{\lambda+1}\right)\delta_{2n}S_2 + k\epsilon(2-n-n^2)F_n\right\}. \quad (5.22)$$

By considering the drop shape relative to axes rotating with an angular velocity $\frac{1}{2}\boldsymbol{\omega}$, and by representing time derivatives with respect to such axes with D/Dt, it is seen that

$$\frac{DF_n}{Dt} = \frac{n(2n+1)}{2(n-1)(2n^2+4n+3)}(\lambda\epsilon)^{-1}\left\{\left(\frac{19\lambda+16}{\lambda+1}\right)\delta_{2n}S_2 + k\epsilon(2-n-n^2)F_n\right\}. \quad (5.23)$$

By using (5.19), (5.20) and (5.21), this further reduces to

$$\frac{D}{Dt}(F_{p_1p_2\ldots p_n}) = -\frac{n(n+2)(2n+1)}{2(2n^2+4n+3)}\left(\frac{k}{\lambda}\right)(F_{p_1p_2\ldots p_n}) \quad \text{for } n > 2, \quad (5.24)$$

and

$$\frac{D}{Dt}(F_{pq}) = \tfrac{5}{19}(\lambda\epsilon)^{-1}\left\{\frac{19\lambda+16}{6(\lambda+1)}e_{pq} - 4k\epsilon(F_{pq})\right\}. \quad (5.25)$$

We note that (5.24), for F_n $(n > 2)$, does not involve the flow field, and that it possesses the solution

$$\left.\begin{aligned} F_{p_1p_2\ldots p_n} &= C_{p_1\ldots p_n}\exp\left\{-\frac{n(n+2)(2n+1)}{2(2n^2+4n+3)}\left(\frac{k}{\lambda}\right)t\right\}, \\ \text{so that} \quad F_n &= C_n\exp\left\{-\frac{n(n+2)(2n+1)}{2(2n^2+4n+3)}\left(\frac{k}{\lambda}\right)t\right\}, \end{aligned}\right\} \quad (5.26)$$

where C_n is a surface harmonic of order n and is in fact the value of F_n at $t = 0$.

Thus, if the sphere starts off in the undeformed state or in a state for which $F_n = 0$ for all $n \geqslant 3$, then it follows from (5.26) that at all future times the only non-zero harmonic F_n is the one of order 2, i.e. $F_n = 0$ for $n \geqslant 3$ for all $t > 0$. However, for an arbitrary initial deformation, it is seen that the harmonics F_n, for $n \geqslant 3$, are transient, dying away in a time of order (λ/k). For this reason, in the following two sections, where examples of drop deformation are given, we will consider drops which are initially spherical so that $F_n = 0$ for all $n \geqslant 3$; the shape of the drop is then spheroidal and given by

$$r = 1 + \epsilon F_{pq}\left[\frac{\partial^2}{\partial r_p\, \partial r_q}(r^{-1})\right]_{r=1}, \tag{5.27}$$

with F_{pq} given by (5.25).

6. Drop in time-dependent shear flow

As an example of the use of the results derived in § 5, we consider a drop placed in a fluid undergoing a time-dependent shear flow given by

$$\mathbf{U} = (0, 0, \gamma(t)\, r_2), \tag{6.1}$$

relative to a set of axes fixed in space, where $\gamma(t)$ is a given function of time t. The vorticity $\boldsymbol{\omega}$ is then

$$\boldsymbol{\omega} = (\gamma(t), 0, 0), \tag{6.2}$$

and the rate-of-strain tensor e_{ij} given by

$$\left.\begin{aligned} e_{23} = e_{32} &= \tfrac{1}{2}\gamma(t), \\ e_{ij} &= 0 \text{ otherwise.}\end{aligned}\right\} \tag{6.3}$$

In order to use (5.25), one must express e_{ij} relative to axes $(\bar{r}_1 \bar{r}_2, \bar{r}_3)$ rotating with angular velocity $\frac{1}{2}\gamma(t)$. The relation between this set of axes and the axes (r_1, r_2, r_3) fixed in space is

$$\left.\begin{aligned} \bar{r}_1 &= r_1, \\ \bar{r}_2 &= r_2 \cos\alpha + r_3 \sin\alpha, \\ \bar{r}_3 &= -r_2 \sin\alpha + r_3\cos\alpha,\end{aligned}\right\} \tag{6.4}$$

where

$$\alpha(t) = \frac{1}{2}\int_0^t \gamma(t')\, dt'. \tag{6.5}$$

If $\bar{e}_{ij}$ and $\bar{F}_{ij}$ are the values of the tensors e_{ij} and F_{ij} relative to the rotating axes $(\bar{r}_1, \bar{r}_2, \bar{r}_3)$, then

$$\left.\begin{aligned} \bar{e}_{22} &= e_{22}\cos^2\alpha + e_{23}\cos\alpha\sin\alpha + e_{32}\sin\alpha\cos\alpha + e_{33}\sin^2\alpha\,, \\ \bar{e}_{23} &= -e_{22}\sin\alpha\cos\alpha + e_{23}\cos^2\alpha - e_{32}\sin^2\alpha + e_{33}\sin\alpha\cos\alpha, \\ \bar{e}_{32} &= -e_{22}\sin\alpha\cos\alpha - e_{23}\sin^2\alpha + e_{32}\cos^2\alpha + e_{33}\sin\alpha\cos\alpha, \\ e_{33} &= +e_{22}\sin^2\alpha - e_{23}\sin\alpha\cos\alpha - e_{32}\sin\cos\alpha + e_{33}\cos{}^2\alpha, \\ \bar{e}_{21} &= e_{21}\cos\alpha\; + e_{31}\sin\alpha, \text{ etc.}\end{aligned}\right\} \tag{6.6}$$

with similar relations between $\bar{F}_{ij}$ and F_{ij}.

Substituting into (6.6) the values of e_{ij} given in (6.3) yields

$$\left.\begin{aligned} \bar{e}_{22} &= \tfrac{1}{2}\gamma \sin 2\alpha, \\ \bar{e}_{33} &= -\tfrac{1}{2}\gamma \sin 2\alpha, \\ \bar{e}_{23} = \bar{e}_{32} &= +\tfrac{1}{2}\gamma \cos 2\alpha, \\ \bar{e}_{ij} &= 0 \quad \text{otherwise.} \end{aligned}\right\} \tag{6.7}$$

Relative to the rotating axes system, the general solution of (5.25) for F_{ij} is

$$\bar{F}_{ij} = \frac{5(19\lambda+16)}{114\lambda(\lambda+1)\epsilon} e^{-(20k/19\lambda)t} \int_{-\infty}^{t} e^{(20k/19\lambda)t'} \bar{e}_{ij}(t')\, dt', \tag{6.8}$$

where the integrand involves the value of e_{ij} at all previous times. From (6.7) and (6.8) it is seen that the only non-zero components of $\bar{F}_{ij}$ are $\bar{F}_{22} = -\bar{F}_{33}$ and $\bar{F}_{23} = \bar{F}_{32}$. Hence F_{ij} relative to the non-rotating axes is given by

$$\left.\begin{aligned} F_{22} &= \bar{F}_{22}\cos^2\alpha + \bar{F}_{33}\sin^2\alpha - \bar{F}_{23}\sin\alpha\cos\alpha - \bar{F}_{32}\sin\alpha\cos\alpha, \\ &= \bar{F}_{22}\cos 2\alpha - \bar{F}_{23}\sin 2\alpha, \\ F_{33} &= \bar{F}_{22}\sin^2 a + \bar{F}_{33}\cos^2\alpha + \bar{F}_{23}\sin\alpha\cos\alpha + \bar{F}_{32}\sin\alpha\cos\alpha, \\ &= -\bar{F}_{22}\cos 2\alpha + \bar{F}_{23}\sin 2\alpha, \\ F_{23} &= \bar{F}_{22}\sin\alpha\cos\alpha - \bar{F}_{33}\sin\alpha\cos\alpha + \bar{F}_{23}\cos^2\alpha - \bar{F}_{32}\sin^2\alpha, \\ &= \bar{F}_{22}\sin 2\alpha + \bar{F}_{23}\cos 2\alpha, \\ F_{32} &= \bar{F}_{22}\sin\alpha\cos\alpha - \bar{F}_{33}\sin\alpha\cos\alpha - \bar{F}_{23}\sin^2\alpha + \bar{F}_{32}\cos^2\alpha, \\ &= \bar{F}_{22}\sin 2\alpha + \bar{F}_{23}\cos 2\alpha, \\ F_{ij} &= 0 \quad \text{otherwise.} \end{aligned}\right\} \tag{6.9}$$

Therefore, by the substitution of $\bar{e}_{ij}$ from (6.7) into (6.8), and using the resulting expressions for $\bar{F}_{ij}$ in (6.9), one obtains

$$\left.\begin{aligned} F_{22} = -F_{33} &= \frac{5(19\lambda+16)}{228\lambda(\lambda+1)\epsilon} e^{-(20k/19\lambda)t} \int_{-\infty}^{t} \gamma(t')\, e^{(20k/19\lambda)t'} \sin 2\{\alpha(t')-\alpha(t)\}\, dt', \\ F_{23} = F_{32} &= \frac{5(19\lambda+16)}{228\lambda(\lambda+1)\epsilon} e^{-(20k/19\lambda)t} \int_{-\infty}^{t} \gamma(t')\, e^{(20k/19\lambda)t'} \cos 2\{\alpha(t')-\alpha(t)\}\, dt'. \end{aligned}\right\} \tag{6.10}$$

All other components of F_{ij} are zero. By the definition of the tensor F_{ij}, the shape of the drop is

$$\begin{aligned} r &= 1 + \epsilon F_{ij}(r^{-1})_{ij} \,.\, r^3, \\ &= 1 + \epsilon[3F_{22}(r_2^2 - r_3^2) + 6F_{23} r_2 r_3]. \end{aligned} \tag{6.11}$$

Defining a set of spherical polar axes (r, θ, ϕ) with the 1-axis as polar axis (figure 1), it is seen that (6.11) may be put in the alternative form

$$r = 1 + D\sin(2\phi + \beta), \tag{6.12}$$

where

$$\left.\begin{aligned} D &= 3\epsilon\sqrt{\{(F_{22})^2 + (F_{23})^2\}}, \\ \tan\beta &= F_{22}/F_{23} \quad [\sin\beta = F_{22}/\sqrt{\{(F_{22})^2 + (F_{23})^2\}}]. \end{aligned}\right\} \tag{6.13}$$

Equation (6.12) represents a spheroid with semi-axes of lengths $1-D$, 1 and $1+D$, the major axis (of length $1+D$) and minor axis (of length, $1-D$) lying in the

2–3 plane, the intermediate axis lying along the 1-axis. If α is the angle the major axis of spheroid makes with the 2-axis. (figure 2), then

$$\alpha = \tfrac{1}{4}\pi - \tfrac{1}{2}\beta; \tag{6.14}$$

the shape of the drop is now

$$r = 1 + D\cos 2(\phi - \alpha). \tag{6.15}$$

If L and B are respectively the length $(2+2D)$ and the breadth $(2-2D)$, then

$$(L-B)/(L+B) = D, \tag{6.16}$$

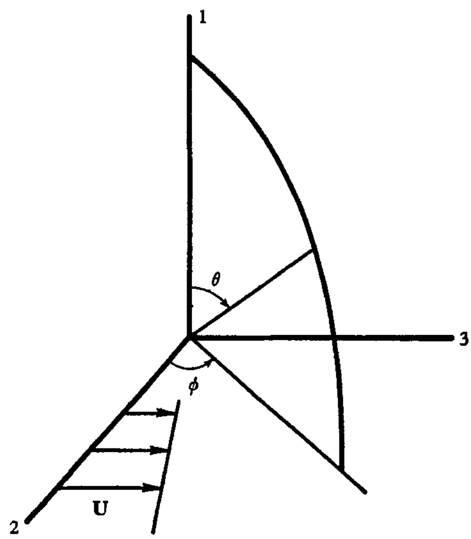

FIGURE 1. Co-ordinate system.

so that the quantity D is the measure of the deformation used by Taylor (1934) and Rumscheidt & Mason (1961).

As a particular case, consider an initially spherical drop placed in a fluid at rest, which at a time $t = 0$ is suddenly given a uniform shearing motion of magnitude unity (in our dimensionless variables). Thus

$$\left.\begin{aligned} \gamma(t) &= 0, \quad \text{if} \quad t < 0, \\ &\ 1, \quad \text{if} \quad t \geqslant 0; \end{aligned}\right\} \tag{6.17}$$

and by (6.5)

$$\left.\begin{aligned} \alpha(t) &= 0, \quad \text{if} \quad t < 0, \\ &\ \tfrac{1}{2}t, \quad \text{if} \quad t \geqslant 0. \end{aligned}\right\} \tag{6.18}$$

By substituting the values of $\gamma(t)$ and $\alpha(t)$ from (6.17) and (6.18) into (6.10) and evaluating the integrals, one obtains values of F_{22}, F_{33}, F_{23} and F_{32} given by

$$\begin{aligned} \epsilon F_{22} = -\epsilon F_{33} &= \frac{5(19\lambda+16)}{12(\lambda+1)\{(20k)^2+(19\lambda)^2\}} \\ &\quad \times [-19\lambda + \sqrt{\{(20k)^2+(19\lambda)^2\}}\, e^{-(20k/19\lambda)t} \cos(t-\bar{\beta})], \\ \epsilon F_{23} = \epsilon F_{32} &= \frac{5(19\lambda+16)}{12(\lambda+1)\{(20k)^2+(19\lambda)^2\}} \\ &\quad \times [+20k + \sqrt{\{(20k)^2+(19\lambda)^2\}}\, e^{-(20k/19\lambda)t} \sin(t-\bar{\beta})], \end{aligned} \tag{6.19}$$

where

$$\bar{\beta} = \tan^{-1}(20k/19\lambda). \tag{6.20}$$

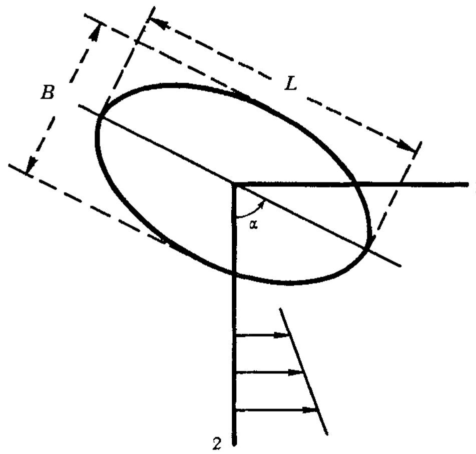

FIGURE 2. Spheroidal drop shape. $D = (L-B)/(L+B)$.

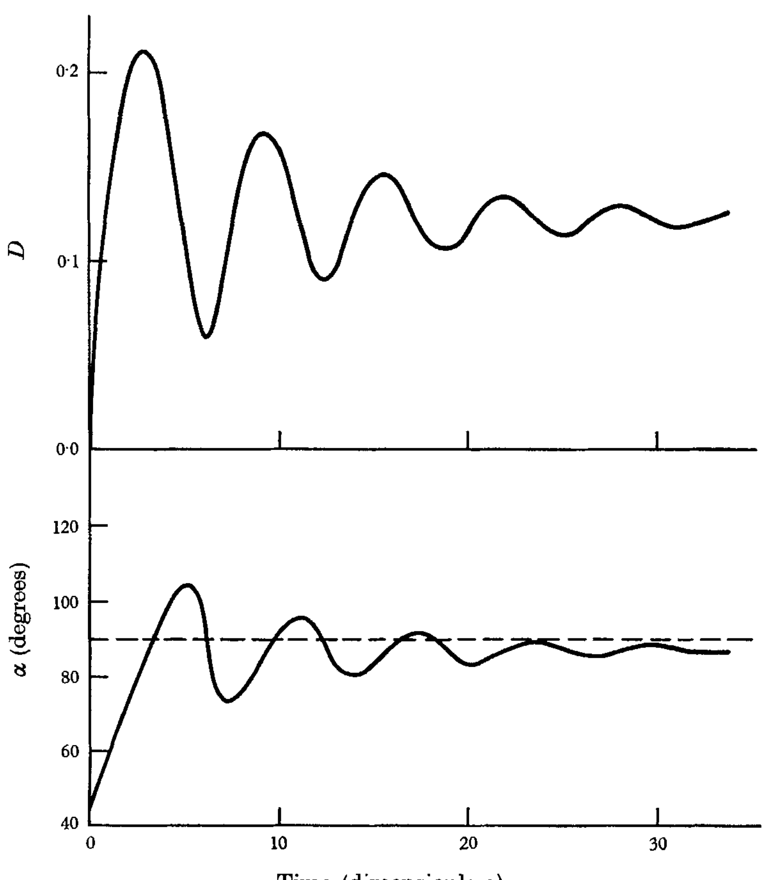

FIGURE 3. Variation of α and D with time t for the case $\lambda = 10$, $k = 1$.

The deformation of the drop given by the quantities D and α is determined by substituting the values of F_{22} and F_{23} from (6.19) into the expressions

$$D = 3\sqrt{\{(\epsilon F_{22})^2 + (\epsilon F_{23})^2\}}, \tag{6.21}$$

$$\alpha = \tfrac{1}{4}\pi - \tfrac{1}{2}\tan^{-1}(F_{22}/F_{23}). \tag{6.22}$$

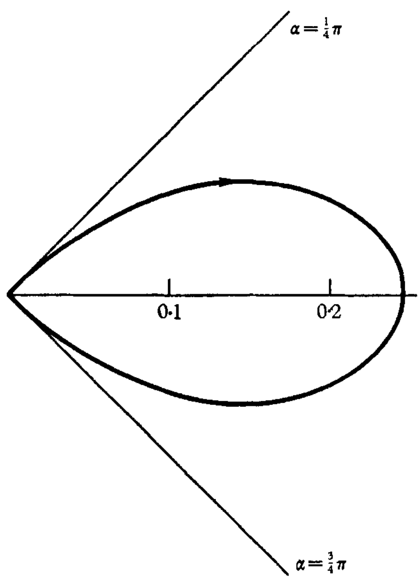

FIGURE 4(*a*). Plot of (D, α) on polar diagram for $\lambda = 10$, $k = 0$.

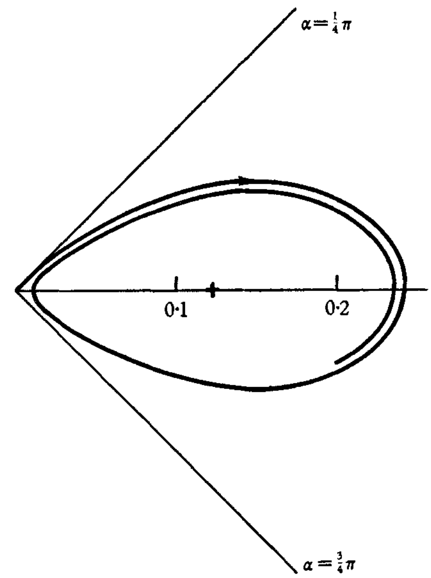

FIGURE 4(*b*). Plot of (D, α) on polar diagram for $\lambda = 10$, $k = 0{\cdot}1$. + represents final equilibrium orientation.

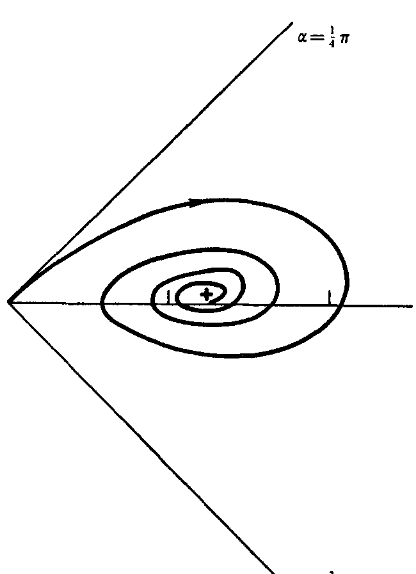

FIGURE 4(*c*). Plot of (D, α) on polar diagram for $\lambda = 10$, $k = 1{\cdot}0$. + represents final equilibrium orientation.

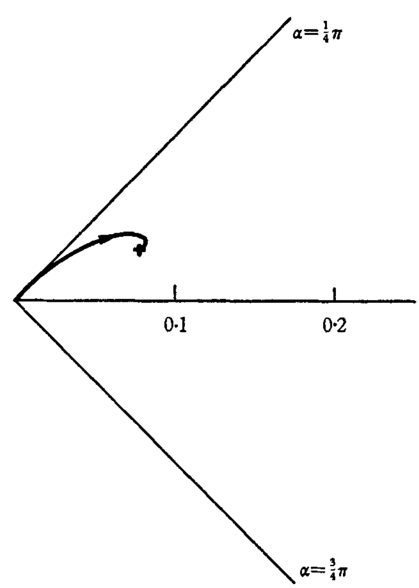

FIGURE 4(*d*). Plot of (D, α) on polar diagram for $\lambda = 10$, $k = 10{\cdot}0$. + represents final equilibrium orientation.

The variation of D and α with time t are shown in figure 3 for the case $\lambda = 10$, $k = 1$, whilst in figures 4(*a*) to 4(*d*) values of D and α are plotted on a polar diagram for $\lambda = 10$ and $k = 0$, 0·1, 1·0 and 10·0.

It is seen that after a long time the drop assumes a steady shape with a deformation given by

$$D = \frac{5(19\lambda+16)}{4(\lambda+1)\sqrt{\{(20k)^2+(19\lambda)^2\}}}, \tag{6.23}$$

$$\alpha = \tfrac{1}{4}\pi + \tfrac{1}{2}\tan^{-1}(19\lambda/20k). \tag{6.24}$$

This steady situation is shown in figure 5, in which lines of constant D and α are plotted on a k^{-1}, λ^{-1} co-ordinate system. The deformation given by (6.19), (6.20) and (6.21) is seen to be small *either* when λ is large, *or* when k is large, *or* when both λ and k are large.

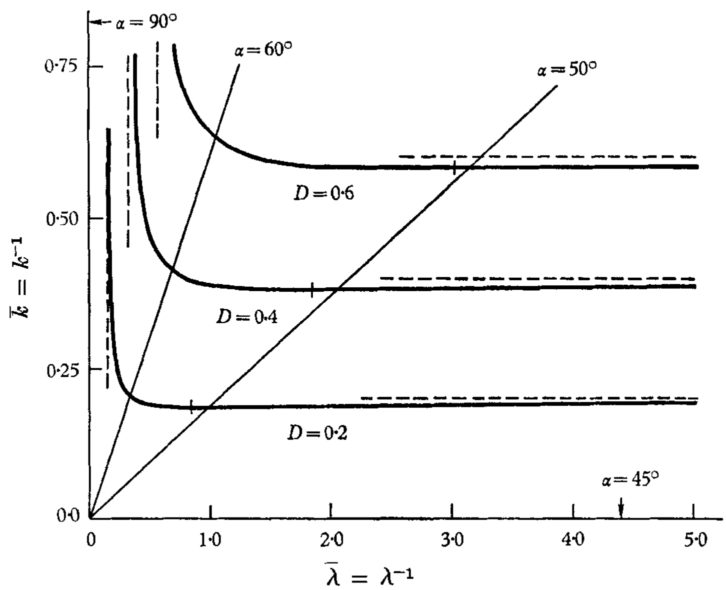

FIGURE 5. Lines of constant D and α for equilibrium orientation in steady shear flow.

For a steady shear flow the steady state deformation is given by (6.25) and (6.24). These reduce to

$$D \sim \frac{19\lambda+16}{16(\lambda+1)}\frac{1}{\bar{k}}, \tag{6.25}$$

$$\alpha \sim \tfrac{1}{4}\pi, \tag{6.26}$$

for the special case of $\lambda = O(1)$ and $k \to \infty$, and to

$$D \sim \tfrac{5}{4}\lambda^{-1}, \tag{6.27}$$

$$\alpha \sim \tfrac{1}{2}\pi, \tag{6.28}$$

for $k = O(1)$ and $\lambda \to \infty$. It is seen that (6.25) and (6.26) agree with the drop shape obtained by Taylor (1934) for a drop in a steady shear flow for the case in which interfacial tension effects are dominant over viscous effects. Again, for the opposite situation, in which interfacial tension effects are negligible, (6.27) and (6.28) are valid and are seen to agree with Taylor's results for that case also. When the interfacial tension σ is zero, k is also zero, and (6.19), which is then valid for $\lambda \gg 1$, becomes

$$\left.\begin{aligned} \epsilon F_{22} &= \frac{5(19\lambda+16)}{228\lambda(\lambda+1)}(\cos t - 1), \\ \epsilon F_{23} &= \frac{5(19\lambda+16)}{228\lambda(\lambda+1)}\sin t. \end{aligned}\right\} \tag{6.29}$$

Hence the deformation is given by

$$\left.\begin{aligned} D &\simeq \tfrac{5}{2}\lambda^{-1}\sin\tfrac{1}{2}t, \\ \alpha &= \tfrac{1}{4}(\pi + t), \end{aligned}\right\} \tag{6.30}$$

which represents an undamped periodic oscillation of the drop. Thus for this case, the drop never actually attains the equilibrium situation given by (6.27) and (6.28), as would be the case for $\lambda \gg 1$ and k small but *non-zero*.

7. Drop in hyperbolic flow

Consider now an initially spherical drop placed in a hyperbolic flow field given by

$$\mathbf{U} = (0, -\mathrm{A}r_2, +\mathrm{A}r_3), \tag{7.1}$$

whose magnitude A is zero for all $t < 0$, and is a constant K for $t \geqslant 0$, i.e.

$$\begin{aligned} A &= 0 \quad \text{for} \quad t < 0, \\ &= K \quad \text{for} \quad t \geqslant 0. \end{aligned} \tag{7.2}$$

Since this flow field possesses zero vorticity, (5.25) may be used directly. The rate-of-strain tensor e_{ij} is

$$e_{22} = -K, \quad e_{33} = +K, \quad e_{ij} = 0 \quad \text{otherwise, for all } t \geqslant 0, \tag{7.3}$$

Thus F_{ij} is given by

$$\left.\begin{aligned} F_{22} &= -F_{33} \neq 0, \\ F_{ij} &= 0 \quad \text{otherwise,} \end{aligned}\right\} \tag{7.4}$$

where F_{22} satisfies

$$\frac{\partial F_{22}}{\partial t} + \frac{20k}{19\lambda}F_{22} = -\frac{5(19\lambda+16)K}{114\lambda\epsilon(\lambda+1)}. \tag{7.5}$$

Hence

$$(\epsilon F_{22}) = -\frac{(19\lambda+16)K}{24(\lambda+1)k}[1 - e^{-(20k/19\lambda)t}]. \tag{7.6}$$

Taking spherical polar axes, as in the previous section, it is seen that the shape of the drop for $t \geqslant 0$ is given by

$$r = 1 + 3\epsilon F_{22}\cos 2\phi$$

or

$$r = 1 - \frac{(19\lambda+16)K}{8(\lambda+1)k}[1 - e^{-(20k/19\lambda)t}]\cos 2\phi. \tag{7.7}$$

This gives a small deformation and is therefore valid *only* when k is large. Unlike the results for a drop in shear flow given in § 6, this result is *not* valid for λ large and $k = O(1)$.

8. Zero interfacial tension

It was shown in § 6 that an initially spherical drop with zero interfacial tension, when placed in shear flow impulsively started from rest, would undergo an undamped periodic oscillation given by (6.30). This particular case will now be examined in detail, and an alternative, more simple argument given for the undamped oscillation.

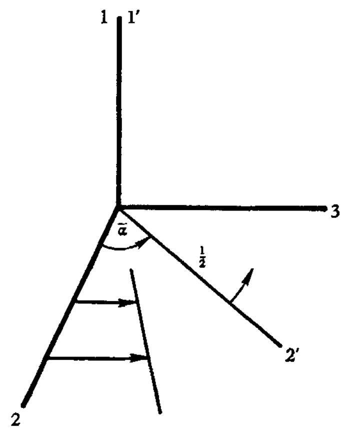

FIGURE 6. System of rotating axes.

Thus we consider a drop which is spherical at time $t = 0$, and placed in a shear flow of magnitude unity, i.e.

$$\mathbf{U} = (0, 0, r_2). \tag{8.1}$$

It is assumed that the interfacial tension σ is zero, and that the viscosity ratio λ is very much greater than unity. In this situation, the fluid within the drop, due to its very large viscosity relative to that of the suspending medium, will undergo a motion which is very approximately a solid rotation of angular velocity $\frac{1}{2}$. Thus axes 1′, 2′, 3′ are taken rotating with an angular velocity of $+\frac{1}{2}$ about the 1-axis and coincident with the axes 1, 2, 3 at the initial instant $t = 0$. If $\bar{\alpha}$ is the angle between the 2 and 2′ axes (see figure 6), then

$$\bar{\alpha} = \tfrac{1}{2}t. \tag{8.2}$$

At $t = 0$, the shear flow $\mathbf{U}$, when taken relative to the rotating axes 1′, 2′, 3′, becomes the hyperbolic flow

$$\mathbf{U} = (0, \tfrac{1}{2}r_3', \tfrac{1}{2}r_2'); \tag{8.3}$$

at later times $t > 0$, this flow field is rotated through an angle $(-\bar{\alpha})$ about 1′ axis. Taylor (1934) gave a formula for the rate of radial extension of a viscous drop in hyperbolic flow, which for our present case reduces to

$$\frac{dr}{dt} = \frac{5}{4\lambda}\cos 2(\phi' + \bar{\alpha} - \tfrac{1}{4}\pi), \tag{8.4}$$

where ϕ' is the azimuthal angle measured from the 2′ axis (see figure 7). Substituting $\bar{\alpha}$ from (8.2) into (8.4), and solving for r with the initial condition that $r = 1$ at $t = 0$, one obtains

$$r = 1+\tfrac{5}{2}\lambda^{-1}\sin\tfrac{1}{2}t\cos(2\phi'-\tfrac{1}{2}\pi+\tfrac{1}{2}t). \tag{8.5}$$

If ϕ is the azimuthal angle measured from the 2-axis, then

$$\phi = \phi'+\bar{\alpha} = \phi'+\tfrac{1}{2}t, \tag{8.6}$$

so that, relative to the non-rotating axes, the shape of the drop is

$$r = 1+\tfrac{5}{2}\lambda^{-1}\sin\tfrac{1}{2}t\cos 2\{\phi-\tfrac{1}{4}(\pi+t)\}. \tag{8.7}$$

Hence, with the notation of § 6, the deformation is given by

$$D = \tfrac{5}{2}\lambda^{-1}\sin\tfrac{1}{2}t, \quad \alpha = \tfrac{1}{4}(\pi+t), \tag{8.8}$$

which is just the result for this particular case obtained from the more general theory (see (6.30)).

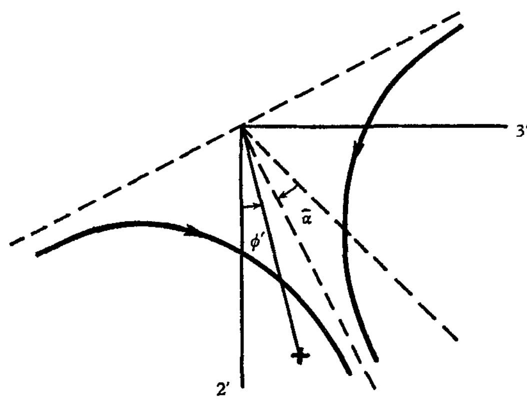

FIGURE 7. Fluid flow relative to rotating axes.

9. Discussion of results

Validity of general results

In the derivation of (5.25), the only restriction placed upon λ and k was that the resulting drop deformation should be small. It was noted in § 6 that the deformation in shear flow was small if *either* λ is large, *or* k is large, *or* λ and k both large, whereas in § 7, the deformation in hyperbolic flow was small *only* if k is large. For a closer examination of the conditions for small deformation, consider the undisturbed flow **U** in the general form (2.4). Such a flow field, when expressed relative to non-rotating axes coincident with principal rate-or-strain axes, takes the dimensional form

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} A_1 & -\Omega_3 & +\Omega_2 \\ +\Omega_3 & A_2 & -\Omega_1 \\ -\Omega_2 & +\Omega_1 & A_3 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}, \tag{9.1}$$

where
$$A_1+A_2+A_3 = 0, \tag{9.2}$$

A_1, A_2 and A_3 are the principal strain rates, and $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ is equal to the fluid angular velocity or one half the vorticity. Relative to non-rotating axes, the time derivative DF_{pq}/Dt appearing in (5.25) may be written as

$$\frac{D}{Dt}(F_{pq}) = \frac{\partial}{\partial t}(F_{pq}) - \tfrac{1}{2}\epsilon_{pjk}\omega_j F_{kq} - \tfrac{1}{2}\epsilon_{qjk}\omega_j F_{pk}.$$

Thus, relative to non-rotating axes, (5.25) is

$$\frac{\partial}{\partial t}(F_{pq}) - \tfrac{1}{2}\epsilon_{pjk}\omega_j F_{kq} - \tfrac{1}{2}\epsilon_{qjk}\omega_j F_{pk} = \tfrac{5}{19}(\lambda\epsilon)^{-1}\left\{\frac{19\lambda+16}{6(\lambda+1)}e_{pq} - 4k\epsilon F_{pq}\right\}. \quad (9.3)$$

Examination of (9.3) shows that the drop deformation is small if for *all* i,

$$\left.\begin{aligned} &\textit{either} & \frac{\mu_0 a}{\sigma}|A_i| \ll 1, \\ &\textit{or} & \frac{\mu_0}{\mu^*}\frac{|A_i|}{\max(|\Omega_k|; k \neq i)} \ll 1. \end{aligned}\right\} \quad (9.4)$$

There is also another reason why, under certain circumstances, the general theory given in §§ 2 to 5 may not give the correct result. It is to be observed that in obtaining the boundary condition (4.10), it is assumed that $\epsilon(\partial f/\partial t)$ is of order ϵ. Thus the theory would not be valid if, for example, $(\partial f/\partial t)$ were of order ϵ^{-1}. This would occur, if there were a very quick change in velocity field $\mathbf{U}$ for cases in which $\lambda = O(1)$ and k is very large. Thus, to cite an example, one notes that for a shear started impulsively from rest, the drop shape, which is given by (6.17) to (6.22), is for $k \gg 1$, $\lambda = O(1)$, given approximately by

$$D = 3\sqrt{\{(\epsilon F_{22})^2 + (\epsilon F_{23})^2\}},$$

$$\alpha = \tfrac{1}{4}\pi - \tfrac{1}{2}\tan^{-1}(F_{22}/F_{23}),$$

where

$$(\epsilon F_{22}) = -\frac{19\lambda+16}{48(\lambda+1)k}e^{-(20k/19\lambda)t}\sin t,$$

$$(\epsilon F_{23}) = +\frac{19\lambda+16}{48(\lambda+1)k}\{1 + e^{-(20k/19\lambda)t}\cos t\}.$$

From this it is observed that, whereas

$$(\epsilon F_{22}) = O\left(\frac{1}{k}\right) \quad \text{and} \quad (\epsilon F_{23}) = O\left(\frac{1}{k}\right),$$

and each is therefore small,

$$\frac{\partial}{\partial t}(\epsilon F_{22}) = O(1) \quad \text{and} \quad \frac{\partial}{\partial t}(\epsilon F_{23}) = O(1),$$

for all times t of order (λ/k). Hence, for small times after the shear is started, one cannot expect the theory to be valid for drops for which $\lambda = O(1)$ and $k \gg 1$. However, it should be noted that even for this case the general theory is valid and may be used for all times $t \gg \lambda/k$.

Drop shape

Under the above conditions, for which the general theory of §§ 2 to 5 is valid, it is seen that for any undisturbed basic flow **U**, the drop possesses a spheroidal shape for all time, if initially it is of spheroidal shape (or in the undeformed spherical shape.) However, if initially it has another shape, then the non-spheroidal part of the deformation is transient, and will die away in a time of order (λ/k).

General flow fields

The shape of a drop in a shear flow started impulsively from rest is given by (6.19) to (6.22), and represents a damped periodic oscillation; the frequency of the oscillation is such that for large λ, one complete oscillation takes place while fluid within the drop undergoes one half a rotation. The damping of this oscillation is proportional to k/λ, or in dimensional quantities to $\sigma/\mu^* Ga$ (G being the value of the shear). Thus one notes that the damping occurs as a result of interfacial tension, the drop undergoing an undamped periodic oscillation for the case of zero interfacial tension (see (6.30)). On the other hand, for a drop in a hyperbolic flow field started impulsively from rest, there is no oscillation. Instead, the drop tends monotonically to an equilibrium shape.

Steady shear flow

The equilibrium shape for a drop in steady shear flow is given by (6.23) and (6.24); it tends in the limits $\lambda = O(1)$, $k \to \infty$ and $k = O(1)$, $\lambda \to \infty$ to the results given by Taylor (1934). One notes that, if λ is large, the drop deformation is small for all values of k, and hence it is of shear G (see figure 5). If one were *slowly* to increase G from zero in this case, the initial deformation would be such that $\alpha = \pi/4$. A further increase in G would cause an increase in both α and deformation D until, for large value of G, $\alpha \to \frac{1}{2}\pi$ and $D \sim \frac{5}{4}\lambda$. This type of behaviour was observed experimentally by Rumscheidt & Mason (1961).

Rheology of a suspension of drops

The general theory given in §§ 2 to 5 forms a very useful first step in a theoretical investigation of the rheological properties of a suspension of drops. It is hoped to investigate this matter further in a future paper.

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